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Rational Approximation with Real, Negative Zeros and Poles

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TO RAJ REDDY AS A TOKEN OF ESTEEM AND AFFECTION

This note is concerned with the approximation of $\cosh\sqrt{x}$ on [0, 1] by polynomials having only real negative zeros and by rational functions having only real negative zeros and poles. We establish here that $\cosh\sqrt{x}$ can be approximated on [0, 1] by polynomials of degree n having only real negative zeros with an error $\leqslant 4n^{-1}$ but not better than $c_1 n^{-1}$ (c_1 some positive constant). Further, we show that $\cosh\sqrt{x}$ cannot be approximated on [0, 1] by rational functions of total degree n having only real negative zeros and poles with an error better than $c_2 n^{-45}$. © 1988 Academic Press, Inc.

INTRODUCTION

Approximation by rational functions having only real negative zeros and poles is a difficult task. The first results in this direction are due to Newman. The only results known so far can be found in [2, 3]. In [2] it has been shown that e^x can be approximated on [0, 1] by polynomials of degree at most n having only real negative zeros with an error $\leqslant 2n^{-1}$ but not better than $(17n)^{-1}$. Further, it has been shown in [2] that e^x can be approximated on [0, 1] by rational functions of total degree at most n having zeros and poles only on the negative real axis with an error $\leqslant n^{-c\log n}$ (c a positive constant) but not better than $(512)^{-n}$. Thus the order of magnitude of the error of best approximation to e^x by polynomials of the above type on [0, 1] is $1/n$. For the case of rational functions, we do not know the corresponding order of magnitude of the error. It is perhaps $e^{-c\sqrt{n}}$ (for some positive constant c). From the well known results of S. N. Bernstein, it follows that in the case of unrestricted polynomial approximation to analytic functions on [0, 1], the degree of convergence of

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best approximation is closely related to the rate of decrease of the Taylor coefficients of the function to be approximated. Now it is natural to ask whether this phenomenon holds also for the case of polynomial approximation with only real negative zeros. On the other hand, it is also natural to ask whether there exist functions whose error in approximation by rational functions, restricted as above, of total degree n does not differ very much from that obtained in approximation by restricted polynomials of degree n. In this connection, we prove here the following:

THEOREM 1.

$$\left\| \cosh \sqrt{x} - \prod_{k=0}^{n} \left(1 + \frac{4x}{(2k+1)^2 \pi^2}\right) \right\|_{L_\infty[0,1]} \leqslant \frac{4}{n}.$$

THEOREM 2. *For every polynomial $p(x)$ of degree n having only real zeros, none $[0, 1]$, we have for all $n \geqslant 1$,*

$$\|\cosh \sqrt{x} - p(x)\|_{L_\infty[0,1]} \geqslant \frac{c_1}{n}.$$

THEOREM 3. *For every rational function $r(x)$ of total degree n having zeros and poles only on the negative real axis, we have for all $n \geqslant 1$,*

$$\|\cosh \sqrt{x} - r(x)\|_{L_\infty[0,1]} \geqslant \frac{c_2}{n^{4.5}}.$$

We need the following

LEMMA[2]. *Suppose a polynomial $p(x)$ is of degree at most n ($\geqslant 1$), has real zeros only, and $p(x) > 0$ on $[a, b]$. Then $[p(x)]^{1/n}$ is concave there.*

Proof of Theorem 1. We have

$$\cosh \sqrt{x} = \sum_{k=0}^{\infty} \frac{x^k}{(2k)!} = \prod_{k=0}^{\infty} \left(1 + \frac{4x}{(2k+1)^2 \pi^2}\right).$$

Now set for $n \geqslant 1$,

$$p(x) = \prod_{k=0}^{n} \left(1 + \frac{4x}{(2k+1)^2 \pi^2}\right).$$

Then

$$\|\cosh \sqrt{x} - p(x)\|_{L_\infty[0,1]} \leqslant p(1) \left[\prod_{k=n+1}^{\infty} \left(1 + \frac{4}{(2k+1)^2 \pi^2}\right) - 1 \right] \leqslant \frac{4}{n}.$$

Proof of Theorem 2. We have

$$\cosh \sqrt{x} = \frac{e^{\sqrt{x}} + e^{-\sqrt{x}}}{2}.$$

Now set for $p(x)$ as in the Theorem,

$$\|\cosh \sqrt{x} - p(x)\|_{L_\infty[0,1]} = \varepsilon. \tag{1}$$

Then for any $d > 0$,

$$\begin{gathered} |p(0)| \geqslant 1 - \varepsilon, \qquad |p(d)| \leqslant \frac{e^{\sqrt{d}} + e^{-\sqrt{d}}}{2} + \varepsilon \leqslant \frac{e^{\sqrt{d}} + e^{-\sqrt{d}}}{2(1-\varepsilon)}, \\ |p(2d)| \geqslant \frac{e^{\sqrt{2d}} + e^{-\sqrt{2d}}}{2} - \varepsilon \geqslant \frac{(1-\varepsilon)(e^{\sqrt{2d}} + e^{-\sqrt{2d}})}{2}. \end{gathered} \tag{2}$$

Now by applying our lemma to the above inequalities, we get

$$2\left(\frac{e^{\sqrt{d}} + e^{-\sqrt{d}}}{2(1-\varepsilon)}\right)^{1/n} \geqslant (1-\varepsilon)^{1/n} + (1-\varepsilon)^{1/n}\left(\frac{e^{\sqrt{2d}} + e^{-\sqrt{2d}}}{2}\right)^{1/n}$$

So

$$\begin{aligned} \frac{2}{(1-\varepsilon)^{2/n}} &\geqslant \left(\frac{2}{e^{\sqrt{d}} + e^{-\sqrt{d}}}\right)^{1/n} + \left(\frac{e^{\sqrt{2d}} + e^{-\sqrt{2d}}}{e^{\sqrt{d}} + e^{-\sqrt{d}}}\right)^{1/n} \\ &= e^{-1/n \log((e^{\sqrt{d}} + e^{-\sqrt{d}})/2)} + e^{1/n \log((e^{\sqrt{2d}} + e^{-\sqrt{2d}})/(e^{\sqrt{d}} + e^{-\sqrt{d}}))} \\ &\geqslant 1 - (1/n)\log\left(\frac{e^{\sqrt{d}} + e^{-\sqrt{d}}}{2}\right) + 1 + (1/n)\log\left(\frac{e^{\sqrt{2d}} + e^{-\sqrt{2d}}}{e^{\sqrt{d}} + e^{-\sqrt{d}}}\right) \\ &\quad + \frac{1}{2n^2}\left[\log\left(\frac{e^{\sqrt{2d}} + e^{-\sqrt{2d}}}{e^{\sqrt{d}} + e^{-\sqrt{d}}}\right)\right]^2. \end{aligned} \tag{3}$$

For the case $\sqrt{2d} = \frac{1}{200}$, it is easy to verify that

$$\log\left[\frac{e^{\sqrt{2d}} + e^{-\sqrt{2d}}}{e^{\sqrt{d}} + e^{-\sqrt{d}}}\right] \geqslant \log\left[\frac{e^{\sqrt{d}} + e^{-\sqrt{d}}}{2}\right]. \tag{4}$$

Hence we have from (3) and (4),

$$\frac{2}{(1-\varepsilon)^{2/n}} \geqslant 2 + \frac{1}{2n^2}\log^2\left(\frac{e^{\sqrt{2d}} + e^{-\sqrt{2d}}}{e^{\sqrt{d}} + e^{-\sqrt{d}}}\right). \tag{5}$$

From (5) we obtain easily

$$\frac{1}{(1-\varepsilon)} \geqslant \left[1+\frac{1}{4n^2}\log^2(\cdot)\right]^{n/2} \geqslant 1+\frac{8}{n}\log^2(\cdot),$$

which implies

$$\varepsilon \geqslant \frac{c_1}{n}.$$

Proof of Theorem 3. We use the following well known formula [1, p. 37 and footnote 5 on p. 8].

$$\sum_{k=0}^{m} (-1)^k \binom{m}{k} \frac{1}{k+s} = \frac{m!}{s(s+1)(s+2)\cdots(s+m)} \qquad (m \geqslant 0). \tag{6}$$

In (6), set $s=m(1+t)$ and integrate. Then we get for $u>0$,

$$\sum_{k=0}^{m} (-1)^k \binom{m}{k} \log\left(1+\frac{mu}{m+k}\right)$$

$$= \frac{1}{\binom{2m}{m}} \int_0^u \frac{dt}{(1+t)(1+mt/(m+1))\cdots(1+mt/(2m))}$$

$$\leqslant \binom{2m}{m}^{-1} \int_0^\infty \frac{dt}{(1+2t/3)^{m+1}} = \frac{3}{2m\binom{2m}{m}}. \tag{7}$$

Set

$$s+k=\sqrt{m^2+mk}, \qquad 0 \leqslant k \leqslant m.$$

Then it is easy to check that

$$\begin{aligned} s=\sqrt{m^2+mk}-k &\leqslant m, & k &\geqslant 0,\\ &\leqslant \frac{3m}{4}, & k &\geqslant \frac{m}{2},\\ &\leqslant \frac{3m}{5}, & k &\geqslant \frac{3m}{4}. \end{aligned}$$

Again (6), with $s+k=\sqrt{m}\sqrt{m+k}$, yields

$$\sum_{k=0}^{m} (-1)^k \binom{m}{k} \frac{1}{s+k} = \frac{m!}{s(s+1)\cdots(s+m)} \geqslant \frac{(1.48)^m}{m\binom{2m}{m}}. \tag{8}$$

Consider now a rational function

$$r(x)=e^{-c}\prod_{k=0}^{n}(1+xu_i)^{\varepsilon_i},\qquad \varepsilon_i=\pm 1,\quad n\geqslant 1$$

and set

$$\|\operatorname{coh}\sqrt{x}-r(x)\|_{L_\infty[0,1]}=\delta. \tag{9}$$

To prove Theorem 3 we can assume that among the rational functions of the type of the Theorem, $r(x)$ is of best uniform approximation to $\cosh\sqrt{x}$ on $[0, 1]$. In view of Theorem 1, $\delta\leqslant 4n^{-1}$. Hence we get from (9) for all $n\geqslant 8$ and for $x\in[0, 1]$,

$$|r(x)|\geqslant\cosh\sqrt{x}-\delta\geqslant 1-\delta\geqslant\tfrac{1}{2}. \tag{10}$$

as $\cosh\sqrt{x}=(e^{\sqrt{x}}+e^{-\sqrt{x}})/2$, have by (9) for all $x\in[0, 1]$,

$$\left|\frac{e^{\sqrt{x}}}{2}-r(x)\right|\leqslant\frac{2\delta+1}{2}. \tag{11}$$

From (10) and (11),

$$\left|\frac{e^{\sqrt{x}}}{2r(x)}-1\right|\leqslant\frac{(2\delta+1)}{2\,|r(x)|}\leqslant 2\left(\frac{1}{2}+\delta\right), \tag{12}$$

$$\left|\frac{e^{\sqrt{x}}}{4r(x)}\right|\leqslant(1+\delta). \tag{13}$$

From (13), one has easily

$$\sqrt{x}-\log 4+c-\sum_{i=0}^{n}\varepsilon_i\log(1+xu_i)\leqslant\log(1+\delta)\leqslant\delta. \tag{14}$$

Set

$$x=\frac{m}{m+k},\qquad k=0, 1, 2, 3, ..., m.$$

Then

$$\sqrt{\frac{m}{m+k}}-\log 4+c-\sum_{i=0}^{n}\varepsilon_i\log\left(1+\frac{mu_i}{m+k}\right)\leqslant\delta. \tag{15}$$

Now by applying the difference operator Δ, m times on both sides of (15), we get in view of (7) and (8),

$$-\frac{3n}{2m}+(1.48)^m \leqslant \binom{2m}{m} 2^m \delta \leqslant 8^m \delta. \tag{16}$$

Set

$$m=\left[\frac{\log n}{\log(1.48)}\right],$$

where $[x]$ denotes the largest integer $\leqslant x$.

Then

$$\frac{\log n}{\log(1.48)}-1 \leqslant m \leqslant \frac{\log n}{\log(1.48)}. \tag{17}$$

A simple manipulation based on (16) and (17) yields

$$-\frac{3n\log(1.48)}{2\log(n/1.48)}+\frac{n}{1.48} \leqslant \delta n^{5.41}. \tag{18}$$

From (18) we get for all $n \geqslant 1$,

$$\delta \geqslant c_2 n^{-4.5}.$$

Hence the theorem is proved.

Remark. From Theorems 1, 2 and from the results of [2] it follows that both $\cosh\sqrt{x}$ and e^x can be approximated on [0, 1] by polynomials of degree at most n having only real negative zeros with an error $\leqslant c_4/n$ but not better than c_5/n, even though $\cosh\sqrt{x}$ is an entire function of order $\frac{1}{2}$ and e^x is an entire function of order 1.

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